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Scalar second-order evolution equations possessing an irreducible sl_2-valued zero-curvature representation

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Abstract
We find all scalar second-order evolution equations that possess an sl_2-valued zero-curvature representation irreducible to a proper subalgebra of sl_2. None of these zero-curvature representations depends on a parameter that could serve as the spectral parameter.

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1. Introduction

For more than 20 years, researchers have been attracted by the problem of classification of nonlinear systems possessing a zero-curvature representation (ZCR). Efforts are focused on ZCRs taking values in a non-solvable Lie algebra $\mathfrak{g}$ and depending on a nonremovable parameter, in expectation that they will be suitable for the Zakharov and Shabat [20] formulation of integrability (*S-integrability*) and lead to soliton solutions. A ZCR does not entail integrability if it lacks the spectral parameter; however, even then it constitutes the basic structure underlying such nonlocal concepts as non-Abelian pseudopotentials and Bäcklund transformations. A whole set of examples of nonparametric ZCRs is provided by Gauss–Mainardi–Codazzi systems (see [17]) of geometry of immersed surfaces. Sakovich [14] related non-parametric ZCRs to continual classes of equations rather than hierarchies.

The problem of finding ZCRs is equivalent to that of computing finite-dimensional linear coverings in the sense of Krasil'shchik and Vinogradov [6], which are often just realizations of the Wahlquist–Estabrook prolongation structures [19]. Widely known computational procedures, which rely on deriving an incomplete set of commutation relations of the unknown Lie algebra $\mathfrak{g}$, may be considered algorithmic only when the order of the ZCR is lower than that of the equation [3, 4]. This is essentially why they are insufficient for solving classification problems, unless in combination with methods based on different criteria of integrability. The most complete lists of integrable systems obtained so far resulted from the formal symmetry approach [9, 10, 13].

In this paper, we instead apply the method of [7, 8], which does not put any restriction on the order of the ZCR, but requires the Lie algebra $\mathfrak{g}$ to be fixed. As a result, we obtain all second-order scalar evolution equations

$$u_t = F(t, x, u, u_x, u_{xx}) \tag{1}$$

that possess an irreducible sl_2-valued zero-curvature representation in the sense of being not reducible to a proper subalgebra of sl_2. We arrive at a single previously unnoticed class of equations parametrized by one function of the coordinates t, x. We also distinguish a particular subclass of equations that admit a single conservation law. None of the ZCRs admits a substantial parameter, hence they are of no relevance to S-integrability.

The exclusion of reducible ZCRs is justified. Recall that maximal subalgebras in sl_2 are conjugated to the subalgebra composed of lower-triangular matrices. A ZCR, with or without a parameter, represents a very different quality if gauge equivalent to a lower triangular one. Not only do lower triangular ZCRs exist for many non-S-integrable equations, such as the Burgers equation, but also they have a rather trivial origin, namely a single 'chain' of conservation laws (see the explanation following definition 2). Calogero and Nucci [2] gave abundant examples of ZCRs with parameters, derived from a single conservation law; these are all reducible to an Abelian subalgebra. Paradoxically enough, all the previously found ZCRs [1, 11] for second-order scalar evolution equations turn out to be reducible, and as such fall outside our classification.

2. Preliminaries

Let E be a nonlinear partial differential equation on a number of functions in two independent variables t, x. Let $\mathfrak{g}$ be a non-solvable matrix Lie algebra. By a $\mathfrak{g}$-*valued* ZCR for E we mean a $\mathfrak{g}$-valued one-form $\alpha = A\,\mathrm{d}x + B\,\mathrm{d}t$ such that

$$\mathrm{d}\alpha = \tfrac{1}{2}[\alpha, \alpha] \tag{2}$$

holds as a consequence of E.

Let G be the connected and simply connected matrix Lie group associated with $\mathfrak{g}$. Then for an arbitrary G-valued function S, the form

$$\alpha^S = \mathrm{d}S S^{-1} + S\alpha S^{-1} \tag{3}$$

constitutes another ZCR, which is said to be *gauge equivalent* to the former. Gauge equivalent ZCRs may be viewed as identical geometric objects (connections). A $\mathfrak{g}$-valued ZCR is said to be *reducible* if it is gauge equivalent to a ZCR taking values in a proper subalgebra of $\mathfrak{g}$; otherwise it is said to be *irreducible*.

Let us give a self-contained description of the general algorithm to compute ZCRs [7, 8] as we use it here. For simplicity we restrict ourselves to a single non-linear nth order evolution equation

$$u_t = F(t, x, u, u_1, \ldots, u_n). \tag{4}$$

Here t, x denote coordinates, u is a single field variable, and $u_1 = u_x$, $u_2 = u_{xx}$, etc, stand for the derivatives, while F is a smooth function of its variables. Let us consider the corresponding equation manifold, which we also denote by E. Namely, let E be the infinite-dimensional space $\mathbb{R}^\infty$ endowed with the coordinates t, x, u and u_k, $k \geqslant 1$ (this is just another way of saying that $t, x, u, u_1, u_2, \ldots$ are considered to be independent quantities). We define smooth functions as functions on E that depend only on a finite number of variables, smoothly in the usual sense. The total derivatives $D_x = \partial/\partial x + u_1\partial/\partial u + \cdots + u_{k+1}\partial/\partial u_k + \cdots$, $D_t = \partial/\partial t + F\partial/\partial u + \cdots + D_x^k F\partial/\partial u_k + \cdots$ then act on smooth functions in a well-defined way: e.g., $u_t = F$, $u_{tx} = D_x F$, $u_{txx} = D_{xx}F, \ldots$, are smooth functions on E.

In these terms, a ZCR for equation (4) is a pair of $\mathfrak{g}$-valued functions A, B on E satisfying equation (2), which may be written as

$$D_t A - D_x B + [A, B] = 0. \tag{5}$$

Let us introduce new operators that act on an arbitrary $\mathfrak{g}$-valued function C on E as follows:

$$\widehat{D}_x C = D_x C - [A, C] \qquad \widehat{D}_t C = D_t C - [B, C]. \tag{6}$$

Operators $\widehat{D}_x$, $\widehat{D}_t$ commute whenever (A, B) is a ZCR. We also set $\widehat{D}_i = \widehat{D}_x \circ \cdots \circ \widehat{D}_x$ (i times).

By [7] for every non-trivial ZCR there exists a non-zero *characteristic matrix* R, which is a $\mathfrak{g}$-valued function defined on E (see also the independent definition by Sakovich [14]). The following proposition is proved in ([7], properties 2.7 and 3.9).

Proposition 1.

(1) The characteristic matrix R for a ZCR of the evolution equation (4) satisfies

$$-\widehat{D}_t R = \sum_i (-\widehat{D})_i \left(\frac{\partial F}{\partial u_i} R\right). \tag{7}$$

(2) Gauge-equivalent ZCRs have conjugate characteristic matrices.

In the sequel we consider a ZCR $A\,\mathrm{d}x + B\,\mathrm{d}t$ taking values in sl_2. We shall write the two sl_2-matrices as

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix} \qquad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix}. \tag{8}$$

Equation (5), the starting point of the standard Wahlquist–Estabrook procedure, is a highly underdetermined system of three equations on the six unknowns $a_1, a_2, a_3, b_1, b_2, b_3$, consistently with the presence of a large gauge group formed by all G-valued functions S on E acting according to formula (3). When augmented with equation (7), the system obtains three new equations and three new unknowns. However, one can limit the number of new unknowns to one by requiring that the characteristic matrix R be in the Jordan normal form. Using the remaining gauge freedom, due to the stabilizer $\mathrm{St}_R \subset G$ of R, one can impose one more constraint on the unknowns a_1, a_2, a_3 (see section 2 for details), to make the whole system determined and suitable for solution.

Reducible ZCRs must be excluded from our classification. A subalgebra of sl_2 to which the ZCR may be reduced, is either an Abelian algebra or the two-dimensional solvable subalgebra representable by lower triangular matrices. Obviously, a ZCR taking values in an Abelian algebra is equivalent to a conservation law (as is the case in [2]) and equation (7) then reduces to the corresponding characteristic equation. In particular, $\widehat{D}_i = D_i$ and the method fails because equation (7) then completely decouples from equation (5). Concerning solvable algebras, the situation is not much different, as we shall see immediately below.

Definition 2. *An sl_2-valued ZCR satisfying the condition $a_2 = b_2 = 0$ is said to be lower triangular.*

For a lower triangular ZCR, it follows from equation (5) that $\phi = a_1\,\mathrm{d}x + b_1\,\mathrm{d}t$ is a conservation law. Let h be the potential of ϕ; then, by the same equation (5), $\phi' = (a_3\,\mathrm{d}x + b_3\,\mathrm{d}t)\,\mathrm{e}^{2h}$ is a conservation law nonlocal over the potential h. This situation will be referred to as a *chain* of conservation laws. Clearly, one can reconstruct the reducible ZCR from the corresponding chain (ϕ, ϕ'). In this way, reducible sl_2-valued ZCRs are equivalent to certain chains of conservation laws. Methods to find them are to be sought among methods to compute nonlocal conservation laws.

Proposition 3. *Let the matrices (8) form a ZCR for the evolution equation (4). Suppose that $a_2 = 0$. Then also $b_2 = 0$ or the ZCR is gauge equivalent to zero.*

Proof. Let us denote by C the matrix (5) evaluated at $a_2 = 0$. By assumption, C is zero on the equation manifold E. If $b_2 \neq 0$, then from the condition $0 = c_2 = -D_x b_2 + 2a_1 b_2$ we compute that $a_1 = \frac{1}{2} D_x b_2/b_2$ on E, and then from the condition $0 = c_1 = D_t a_1 - D_x b_1 - a_3 b_2$ we compute that $a_3 = \frac{1}{2} D_{tx} b_2 \big/ b_2^2 - \frac{1}{2} D_t b_2 D_x b_2 \big/ b_2^3 - D_x b_1/b_2$ on E. Let us introduce a function G on E by the requirement $b_3 = -b_1^2 \big/ b_2 - D_t b_1/b_2 + b_1 D_t b_2 \big/ b_2^2 + \frac{1}{2} D_{tt} b_2 \big/ b_2^2 - \frac{3}{4} (D_t b_2)^2 \big/ b_2^3 + G/b_2$ (recall that $b_2 \neq 0$). Then $0 = c_3 = -D_x G/b_2$, which in the case of an evolution equation implies that G is a function of t only. Then, under the above substitutions for a_1, a_3, b_3, the gauge matrix

$$\begin{pmatrix} b_2^{-1/2} & 0 \\ -\frac{1}{2} D_t b_2 b_2^{-3/2} + b_1 b_2^{-1/2} & b_2^{1/2} \end{pmatrix}$$

sends A to zero and B to

$$\begin{pmatrix} 0 & 1 \\ G & 0 \end{pmatrix}$$

which depends on t at most. The last matrix is sent to zero by gauge transformation with the gauge matrix composed of independent solutions of the equation $s_{tt} = Gs$. $\square$

3. The classification

Let us consider a second-order evolution equation (1) along with the sl_2-matrices A, B satisfying equation (5) but not reducible to a solvable subalgebra. We also assume that $\partial F/\partial u_{xx} \neq 0$. Following [8], we consider the two cases distinguished by their Segre characteristics separately.

3.1. The nilpotent case

Under the notation (8), the Jordan form for R corresponds to $r_1 = 0$, $r_2 = 0$, $r_3 = 1$. The normal form for A, obtained in [8], is given by the single requirement $a_1 = 0$. Indeed, whenever $a_2 \neq 0$ (otherwise the ZCR is either lower triangular or trivial by proposition 3), then one can set $a_1 = 0$ in a general matrix A by means of the gauge matrix

$$\begin{pmatrix} 1 & 0 \\ a_1/a_2 & 1 \end{pmatrix}$$

from the stabilizer of R.

Equation (7) then reduces to the system $T_i = 0$, $i = 1, 2, 3$, where

$$\begin{aligned}
T_1 &:= 2D_x \left(\frac{\partial F}{\partial u_{xx}} \right) a_2 + \frac{\partial F}{\partial u_{xx}} D_x a_2 - \frac{\partial F}{\partial u_x} a_2 + b_2 \\
T_2 &:= 2 \frac{\partial F}{\partial u_{xx}} a_2^2 \\
T_3 &:= -D_{xx} \frac{\partial F}{\partial u_{xx}} + D_x \frac{\partial F}{\partial u_x} - 2 \frac{\partial F}{\partial u_{xx}} a_2 a_3 - \frac{\partial F}{\partial u} - 2b_1.
\end{aligned}$$

Then $a_2 = 0$ by the second equation and $b_2 = 0$ by the first equation, whence the ZCR is lower triangular. Consequently, this case is void in our classification.

3.2. The semisimple case

It will be convenient to change the notation for sl_2-matrices to

$$A = \begin{pmatrix} a_1 & a_2 + a_3 \\ a_2 - a_3 & -a_1 \end{pmatrix}.$$

The Jordan form for R has $r_2 = r_3 = 0$ with $r := r_1$ arbitrary. Unlike in [8], we choose the normal form for A characterized by the single requirement $a_3 = 0$. And indeed, whenever $a_2 + a_3 \neq 0$, which is irrestrictive by proposition 3, one can set a_3 to zero by a gauge transformation from the stabilizer of R. The relevant gauge matrix is diagonal with the diagonal entries h and $1/h$, where $h = ((a_2 - a_3)/(a_2 + a_3))^{1/4}$.

Equations (5) and (7) then assume the form $S_i = 0 = T_i$, $i = 1, 2, 3$, with

$$\begin{aligned}
S_1 &= -D_t a_1 + D_x b_1 + 2a_2 b_3 \\
S_2 &= -D_t a_2 + D_x b_2 - 2a_1 b_3 \\
S_3 &= D_x b_3 + 2a_2 b_1 - 2a_1 b_2 \\
T_1 &= -D_t r - \frac{\partial F}{\partial u} r + D_x\left(\frac{\partial F}{\partial u_x} r\right) - D_{xx}\left(\frac{\partial F}{\partial u_{xx}} r\right) - 4\frac{\partial F}{\partial u_{xx}} r a_2^2 \\
T_2 &= -b_3 + 2\frac{\partial F}{\partial u_{xx}} a_1 a_2 \\
T_3 &= -b_2 + \frac{\partial F}{\partial u_x} a_2 - 2D_x\left(\frac{\partial F}{\partial u_{xx}}\right) a_2 - 2\frac{D_x r}{r}\frac{\partial F}{\partial u_{xx}} a_2 - \frac{\partial F}{\partial u_{xx}} D_x a_2.
\end{aligned} \tag{9}$$

If $a_2 = 0$, then we have $b_2 = b_3 = 0$ by the last two equations and the ZCR reduces to a single conservation law. Therefore, we assume that $a_2 \neq 0$ in the sequel.

Proposition 4. *As solutions to equations (9), functions r, a_1, a_2, b_3 cannot depend on coordinates other than t, x, u, u_x, u_{xx}, whereas functions b_1, b_2 cannot depend on coordinates other than $t, x, u, u_x, u_{xx}, u_{xxx}$.*

Proof. We may assume, without loss of generality, that the functions r, a_1, a_2, b_3 depend on $t, x, u, \ldots, u_k$ and the functions b_1, b_2 depend on $t, x, u, \ldots, u_k, u_{k+1}$ for some $k \geqslant 2$. We perform a downward induction, each step of which consists in deriving appropriate differential consequences of the system (9). Thus, let $k > 2$. Then we have

$$0 = \frac{\partial T_1}{\partial u_{k+2}} = -2\frac{\partial F}{\partial u_{xx}}\frac{\partial r}{\partial u_k}$$

but $\partial F/\partial u_{xx} \neq 0$, whence r does not depend on u_k. Then, similarly,

$$\begin{aligned}
0 &= \frac{\partial S_2}{\partial u_{k+2}} - \frac{\partial T_3}{\partial u_{k+1}} = 2\frac{\partial b_2}{\partial u_{k+1}} \\
0 &= \frac{\partial S_2}{\partial u_{k+2}} + \frac{\partial T_3}{\partial u_{k+1}} = -2\frac{\partial F}{\partial u_{xx}}\frac{\partial a_2}{\partial u_k}
\end{aligned}$$

whence b_2 does not depend on u_{k+1} and a_2 does not depend on u_k. Finally,

$$0 = -2a_2 \frac{\partial S_1}{\partial u_{k+2}} + \frac{\partial S_3}{\partial u_{k+1}} + \frac{\partial T_2}{\partial u_k} = 4a_2 \frac{\partial F}{\partial u_{xx}} \frac{\partial a_1}{\partial u_k}$$
$$0 = 2a_2 \frac{\partial S_1}{\partial u_{k+2}} + \frac{\partial S_3}{\partial u_{k+1}} + \frac{\partial T_2}{\partial u_k} = 4a_2 \frac{\partial b_1}{\partial u_{k+1}}$$
$$0 = -2a_2 \frac{\partial S_1}{\partial u_{k+2}} + \frac{\partial S_3}{\partial u_{k+1}} - \frac{\partial T_2}{\partial u_k} = 2 \frac{\partial b_3}{\partial u_k}$$

whence a_1, b_3 do not depend on u_k and b_1 does not depend on u_{k+1} (recall that $a_2 \neq 0$). This completes the induction step. □

Under the restrictions established in proposition 4, the determining system (9) becomes an overdetermined system of partial differential equations. As such, it can be solved routinely, but its solution is troublesome even with the employment of software capable of automating the derivation of differential consequences. The reason is that the class of second-order evolution equations is invariant with respect to a large group of contact transformations $\bar{x} = \bar{x}(t, x, u, u_x)$, $\bar{u} = \bar{u}(t, x, u, u_x)$, $\bar{t} = \bar{t}(t)$. Below we shall apply a series of suitably chosen contact transformations to achieve substantial reduction of the matrix A.

Proposition 5. *For every second-order evolution equation (1) possessing an irreducible sl_2-valued ZCR there exists a contact transformation such that the transformed a_2 depends on t, x, u, u_x at most.*

Proof. Let functions r, a_i, b_i depend on the coordinates $t, x, u, u_x, u_{xx}, u_{xxx}$ as found in proposition 4. Taking successively the derivatives $\partial S_1/\partial u_{xxxx}$, $\partial S_2/\partial u_{xxxx}$, T_2, $\partial T_3/\partial u_{xxx}$, $\partial T_1/\partial u_{xxxx}$, $\partial S_3/u_{xxx}$, $\partial T_3/\partial u_{xxx}$, $\partial^2 S_1/\partial u_{xxx}^2$, $\partial^2 S_2/\partial u_{xxx}^2$ one may check routinely that

$$\frac{\partial^2 a_2}{\partial u_{xx}^2} = 0 \qquad \text{and} \qquad a_1 \frac{\partial a_2}{\partial u_{xx}} - \frac{\partial a_1}{\partial u_{xx}} a_2 = 0$$

are among differential consequences of system (9). Hence,

(a) a_2 is linear in u_{xx}, i.e., $a_2 = a_{21}(t, x, u, u_x) u_{xx} + a_{20}(t, x, u, u_x)$;
(b) the ratio a_1/a_2 does not depend on u_{xx}.

Now, if $a_{21} = 0$, then the statement is proved. Otherwise, let f_1, f_2 be two functionally independent solutions of the linear equation

$$-\frac{a_{20}}{a_{21}} \frac{\partial f}{\partial u_x} + u_x \frac{\partial f}{\partial u} + \frac{\partial f}{\partial x} = 0. \tag{10}$$

In particular, both f_1 and f_2 do depend on u_x. Then $\bar{t} = t$, $\bar{x} = f_1$, $\bar{u} = f_2$ and $\bar{u}_{\bar{x}} = (\partial f_2/\partial u_x)/(\partial f_1/\partial u_x)$ satisfy the well-known necessary conditions of being a contact transformation:

$$\frac{\dfrac{\partial \bar{u}}{\partial u_x}}{\dfrac{\partial \bar{x}}{\partial u_x}} = \bar{u}_{\bar{x}} = \frac{\dfrac{\partial \bar{u}}{\partial x} + u_x \dfrac{\partial \bar{u}}{\partial u}}{\dfrac{\partial \bar{x}}{\partial x} + u_x \dfrac{\partial \bar{x}}{\partial u}}.$$

Under this transformation, $A\,\mathrm{d}x + B\,\mathrm{d}t$ becomes $\bar{A}\,\mathrm{d}\bar{x} + \bar{B}\,\mathrm{d}\bar{t}$ with $\mathrm{d}\bar{x} = D_x\bar{x}\,\mathrm{d}x + D_t\bar{x}\,\mathrm{d}t$, $\mathrm{d}\bar{t} = \mathrm{d}t$, so that

$$A = \bar{A} D_x \bar{x} = \left(\frac{\partial f_2}{\partial x} + u_x \frac{\partial f_2}{\partial u} + u_{xx} \frac{\partial f_2}{\partial u_x} \right) \bar{A} = \frac{a_2}{a_{21}} \frac{\partial f_2}{\partial u_x} \bar{A}$$

where we have used equation (10). Hence

$$\bar{A} = \frac{a_{21}}{\partial f_2/\partial u_x} \begin{pmatrix} a_1/a_2 & 1 \\ 1 & -a_1/a_2 \end{pmatrix}$$

which is independent of u_{xx}, hence of $\bar{u}_{xx}$, by virtue of statement (b) above. □

Theorem 6. *Every second-order scalar evolution equation (1) possessing an irreducible sl_2-valued ZCR is transformable to an equation of the form*

$$u_t = \frac{\partial\beta}{\partial x}u^2u_{xx} + 2\frac{\partial^2\beta}{\partial x^2}u^2u_x + 4\beta u_x + \left(\frac{\partial^3\beta}{\partial x^3} - 4\frac{\partial\beta}{\partial x}\right)u^3 - 4\frac{\partial\beta}{\partial x}u \tag{11}$$

through a contact transformation. Here β is an arbitrary function of t, x with $\partial\beta/\partial x \neq 0$. The ZCR is then $A\,\mathrm{d}x + B\,\mathrm{d}t$ with

$$A = \begin{pmatrix} \dfrac{1}{u} & 1 \\ 1 & -\dfrac{1}{u} \end{pmatrix} \qquad B = \begin{pmatrix} -\dfrac{\partial\beta}{\partial x}u_x + 4\dfrac{\beta}{u} - \dfrac{\partial^2\beta}{\partial x^2}u & 4\beta + 2\dfrac{\partial\beta}{\partial x}u \\ 4\beta - 2\dfrac{\partial\beta}{\partial x}u & \dfrac{\partial\beta}{\partial x}u_x - 4\dfrac{\beta}{u} + \dfrac{\partial^2\beta}{\partial x^2}u \end{pmatrix}. \tag{12}$$

Proof. Following proposition 5, we assume that the matrix A depends on t, x, u, u_x at most. One may check routinely that

$$\frac{\partial^2 a_2}{\partial u_x^2} = 0 \qquad \text{and} \qquad \frac{\partial^2 a_2}{\partial x\partial u_x} + u_x\frac{\partial^2 a_2}{\partial u\partial u_x} - \frac{\partial a_2}{\partial u} = 0$$

are among differential consequences of the system (9). The general solution is $a_2 = \partial h/\partial x + u_x\partial h/\partial u = D_x h$ for a suitable function $h(t, x, u)$. If a_2 does depend on u_x, then $\partial h/\partial u \neq 0$, whence $\bar{t} = t$, $\bar{x} = h$, $\bar{u} = x$ is a point transformation. If a_2 does not depend on u_x, then h does not depend on u, but does depend on x (otherwise $a_2 = 0$), and $\bar{t} = t$, $\bar{x} = h$, $\bar{u} = u$ is a point transformation. In both cases $A = \bar{A}D_x h = \bar{A}a_2$, whence $\bar{a}_2 = 1$ in the transformed matrix $\bar{A}$.

With $a_2 = 1$, one can check routinely that $\partial a_1/\partial u_x = 0$ is among differential consequences of system (9). If moreover $\partial a_1/\partial u = 0$, then A is completely independent of u and its derivatives, and then so is B, whence the ZCR is gauge equivalent to zero. Therefore, we shall continue with $\partial a_1/\partial u \neq 0$. Then we can apply a point transformation $\hat{x} = \bar{x}$, $\hat{u} = 1/a_1$, which sends a_1 to $1/\hat{u}$ (this choice prevents terms quadratic in u_x from appearing on the right-hand side of equation (1)). It is then a matter of routine to compute all possible forms of the right-hand side F of equation (1) and also the corresponding matrices B. □

There seems to be no earlier appearance of the class (11) in the literature, let alone its 'simplest' member $u_t = u^2u_{xx} + 4xu_x - 4u^3 - 4u$.

The results would be incomplete if we do not establish irreducibility of the ZCR (12). Since reducibility implies existence of at least one local conservation law, we shall start with the following result.

Proposition 7. *Within the class (11), the only equations to possess a conservation law are those with*

$$\beta = \frac{1}{8}\frac{p_t\,\mathrm{e}^{2x} + q_t\,\mathrm{e}^{-2x}}{p\,\mathrm{e}^{2x} - q\,\mathrm{e}^{-2x}} \tag{13}$$

where p, q are arbitrary functions of t such that $(pq)_t \neq 0$. In all these cases the equation has a single conservation law

$$D_t\frac{p\,\mathrm{e}^{2x} + q\,\mathrm{e}^{-2x}}{u} = D_x\left(\frac{1}{2}\frac{(pq)_t(p\,\mathrm{e}^{2x} + q\,\mathrm{e}^{-2x})}{(p\,\mathrm{e}^{2x} - q\,\mathrm{e}^{-2x})^2}u_x + \frac{1}{2}\frac{(p_t\,\mathrm{e}^{2x} + q_t\,\mathrm{e}^{-2x})(p\,\mathrm{e}^{2x} + q\,\mathrm{e}^{-2x})}{(p\,\mathrm{e}^{2x} - q\,\mathrm{e}^{-2x})}\frac{1}{u}\right.$$
$$\left. - \frac{(pq)_t(3p^2\,\mathrm{e}^{4x} + 2pq + 3q^2\,\mathrm{e}^{-4x})}{(p\,\mathrm{e}^{2x} - q\,\mathrm{e}^{-2x})^3}u\right). \tag{14}$$

Proof. A routine computation shows that any characteristics ψ of a conservation law depends on t, x, u at most and satisfies the equations

$$\frac{\partial^2\psi}{\partial x^2} - 4\psi = 0 \qquad \frac{\partial\psi}{\partial u} + 2\frac{\psi}{u} = 0 \qquad \frac{\partial\psi}{\partial t} - 4\beta\frac{\partial\psi}{\partial x} = 0.$$

The rest is easy. □

Another computation shows that for none of the equations of the class (13) the corresponding ZCR (12) can be reduced to the lower triangular form with multiples of (14) on the diagonal. Thus, the ZCRs (12) are indeed irreducible.

It is not obvious that our ZCRs do not depend on any spectral parameter since we fixed the matrix A during computation. A relatively inexpensive test, repeating the computation with fixed β instead of fixed A, actually confirms the absence of a parameter, but cannot rule out parameters present in a larger Lie algebra. A convincible proof of non-integrability consists in checking the criteria used by Svinolupov and Sokolov [15, 16], with the outcome that already the first of them fails.

Finally, a remark on equations determining pseudospherical surfaces (PSS equations) is due. In anticipation of finding new S-integrable nonlinear systems, a number of attempts have been made to classify equations describing pseudospherical surfaces (PSS equations) (see [18], and references therein). Even though being a PSS equation is equivalent to possessing an sl_2-valued ZCR, the classification of second-order scalar evolution PSS equations as obtained by Reyes [11] (see also [5, 12]) has no intersection with ours. This seeming paradox is easily resolved. Each of the ZCRs found by Reyes is reducible to the lower triangular form (the generalized Burgers equation) or even to a single conservation law (the other equations), which are disregarded in our classification. On the other hand, we already saw that equations (11) do not enter the Svinolupov and Sokolov [15, 16] classification, which was the starting point of the Reyes work.

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